Risky Business and the Process of Development*

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Abstract

Risk is an important factor that affects investment decisions, especially for undiversified entrepreneurs in less developed economies. Yet standard macro models of financial frictions do not incorporate risk: short-term returns are known in advance, and investment is fully reversible. Thus, even if entrepreneurs are risk averse and credit constrained, they will invest all of their assets in the firm, until the marginal product of capital equals the interest rate. As a result, standard models often find that productive entrepreneurs quickly save their way out of credit constraints, limiting the effect of financial frictions on output and aggregate productivity. We incorporate risk into a model of financial frictions, by making investment partially irreversible. Productive entrepreneurs accumulate capital substantially more slowly than in the first-best, leading to a reduction in aggregate productivity. Credit can play a role in undoing these frictions if firms have an option to default. Default creates a state-contingent contract, in which the entrepreneur repays if productivity stays high and defaults if productivity falls; this encourages investment and improves welfare through risk-sharing with the bank.

Keywords: Misallocation, Financial Frictions, Irreversible Investment, Risk, Entrepreneurs, Development

JEL: D24, D25, D53, E44, L26, O11, O16

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1 Introduction

How does finance affect development? Economists have studied this question both empirically and theoretically. Empirical work has found evidence for high returns to capital in developing countries (de Mel et al., 2008; Fafchamps et al., 2014; McKenzie, 2017), as well as substantial dispersion in the returns to capital (Banerjee and Duflo, 2005; Hsieh and Klenow, 2009; Hussam et al., 2022; Beaman et al., 2023; Crépon et al., 2023; Hughes and Majerovitz, 2023). Moreover, economists have exploited variation over time and across sectors to find evidence that finance promotes development (Rajan and Zingales, 1998; Bau and Matray, 2023; Sraer and Thesmar, 2023).

In contrast, dynamic models of financial frictions generally have trouble delivering these facts. This is highlighted succinctly in Banerjee and Moll (2010), who show that in dynamic models with credit constraints, firms quickly save their way out of credit constraints, as long as production functions are concave. Even with non-concavities in the production function, these models cannot deliver persistent dispersion in returns to capital across firms. Later work has shown that an environment with productivity shocks can create steady-state dispersion in productivity, with the degree of dispersion governed by the persistence of productivity (Buera and Shin, 2011; Moll, 2014). However, in practice productivity is fairly persistent, and so calibrated dynamic models of financial frictions find modest output losses from financial frictions (Buera et al. 2011; Midrigan and Xu 2014; see Buera et al. 2015 for a review of this literature).

In this paper, we show that these powerful save-your-way-out dynamics are driven by a lack of investment risk. In these models, entrepreneurs know their productivity at the time of investment, and face frictionless rental markets for capital. Thus the entrepreneur knows exactly the return on capital she will get from her investment, and can reverse her investments instantly if her productivity falls. As a result, an entrepreneur that faces a credit constraint will put all of her assets into her business, as long as the marginal product of capital at her firm is less than the interest rate she faces on borrowing or saving. This corner solution means that constrained firms will expand rapidly, limiting the losses from financial frictions.

To break these save-your-way-out dynamics, we introduce introduce investment risk by making investment (partially) irreversible. With irreversible investment, the entrepreneur must consider not just her (known) productivity at the time of investment, but also her unknown future productivities over the lifetime of the investment. If her productivity falls after she invests, she is stuck at that level of capital until depreciation brings her capital back down to the optimal level. This makes investment risky: it will pay off if productivity stays high, but could be a mistake if productivity falls. Moreover, the low productivity state of the world is exactly the state in which the entrepreneur has low consumption and thus high marginal utility.

Since entrepreneurs are risk averse in our model, the addition of partial irreversibility leads to lower levels of investment, as in Angeletos (2007). This also slows down the investment dynamics substantially. Since undiversified entrepreneurs are concerned about risk, they will only begin to equate their expected returns to capital with the interest rate on savings as they accumulate enough assets to self-insure against investment risk. This process is much slower than in models without investment risk. Not only must the entrepreneur accumulate enough assets to finance her firm's optimal level of capital, she also must accumulate a large buffer of assets to insure her own consumption against risk.[1]

These dramatically slower investment dynamics also mean that our model can deliver the empirical facts that prior dynamic models could not. Investment risk results in high returns to capital in equilibrium. Moreover, the slow investment dynamics in our model means that our model delivers substantial dispersion in returns. These high and dispersed returns to capital imply substantial losses from financial frictions.

The inefficiencies highlighted by our model come from a lack of risk-sharing, rather than directly coming from credit constraints. This market incompleteness could be alleviated through a variety of financial contracts, such as equity or insurance. To create a role for credit, we introduce the option to default into our model.

In an otherwise efficient model, default would introduce only distortions. Banks must charge higher interest rates to borrowers in order to account for the possibility of default, which lowers investment at firms with a low probability of default. Moreover, default is costly: in our model, the bank repossesses the firm's capital and liquidates it, which is less efficient than operating the firm at a reduced scale. After a default, entrepreneurs are punished for a period of time by not being allowed to borrow, which imposes further losses.

However, allowing for default turns credit into a state-contingent contract: the entrepreneur does not pay back the loan in the worst states of the world. This makes investment less risky, and encourages more investment. Thus, credit plays a risk-sharing role, and can increase investment, output, and welfare.

The rest of the paper proceeds as follows. Section 2 lays out our dynamic model with and without default, and contrasts this with the planner's problem. Section 3 explains how we calibrate the model, based on values estimated in the empirical literature. Section 4

[1] Formally, an investor with CARA utility will invest a constant amount in a risky asset, while an investor with CRRA utility invests a constant share of her wealth, since her coefficient of absolute risk aversion falls with her wealth. Our entrepreneurs have CRRA utility, so they invest more in their firm as they grow wealthier.

explores firm behavior in the model, highlighting the slowdown of investment dynamics that are crucial to our results. Section 5 uses our calibrated model to study the equilibrium effects of financial frictions. Section 6 concludes.

2 Model

Time is continuous and individuals live indefinitely. In this economy, individuals supply labor to the labor market and operate their own business at the same time. Their entrepreneurial productivity is denoted by z_t, which is the only source of risk. We assume z_t follows a Poisson process. With intensity λ_z, individuals' productivty are reset and redrawn from a given distribution $G(z)$. The intensity controls the persistence of the z_t process (inversely). We assume that $G(z)$ is a discrete distribution. The individual production function is $y_t = z_t k_t^{\alpha} l_t^{\beta}$, where k is capital and l is labor. There are two assets. One is capital, which is subject to adjustment costs. The other is bonds. If they have positive bond holdings, the net interest rate is r_s. If they issue bonds, subject to a collateral constraint, the interest rate is r_b, with $r_b \geq r_s$. We consider two cases. In one, all bonds are risk free, and individuals cannot default on their debt. In the other, individuals can default on their debt.

We can write down the individuals' problem as follows.

$$\begin{gathered}
\max_{c_t, i_t, \tau} \ \mathbb{E}_0 \int_0^{\tau} e^{-\rho t} \frac{c_t^{1-\sigma}}{1-\sigma} dt + e^{-\rho \tau} V^{def}(a=0, k=0, z_t) \\
da = \big(\pi(k_t, z_t) + w + r_a a_t - c_t - i_t - \Phi(i_t, k_t)\big) dt \\
dk = (i_t - \delta k_t) dt \\
a_t \geq -\lambda k_t, \quad k_t \geq 0 \\
\pi(k_t, z_t) \equiv \max_{l} z_t k_t^{\alpha} l_t^{\beta} - wl.
\end{gathered}$$

The default decision is given as an optimal stopping time τ with the continuation value V^{def}, with the assumption that all debt is discharged ($a = 0$) upon default but the defaulting individuals lose their capital ($k = 0$). In the no-default case, τ is not a choice variable at set to $\tau = \infty$. The interest rate r_a in the budget constraint is r_s if $a \geq 0$ and r_b otherwise. The term $\Phi(i_t, k_t)$ is the adjustment costs, which depends on investment i_t and capital stock k_t. Borrowing ($a < 0$) is limited by a simple collateral constraint λk_t.

The problem of an individual in default is as follows.

$$\max_{c_t, i_t} \mathbb{E}_0 \int_0^T e^{-\rho t} \frac{c_t^{1-\sigma}}{1-\sigma} dt + e^{-\rho T} V(a_T, k_T, z_T)$$
$$da = \big(\pi(k_t, z_t) + w + r_s a_t - c_t - i_t - \Phi(i_t, k_t)\big) dt$$
$$dk = (i_t - \delta k_t) dt$$
$$a_t \geq 0, \quad k_t \geq 0$$

Time T is an exogenous random variable, at which point the individual's default record is expunged and he regains full access to the financial markets. The arrival of T is governed by a Poisson process with intensity λ_d. Until then, the individual cannot borrow ($a \geq 0$) but can accumulate bonds and capital, the latter subject to the same adjustment costs.

We now specify the adjustment cost function Φ. We introduce partial irreversibility, by assuming that individuals get back $\phi \leq 1$ for one unit of capital sold. To be precise, we assume the following functional form.

$$\Phi(i, k) = \begin{cases} \frac{\kappa}{2} \left[\frac{i}{k + \bar{k}} \right]^2 (k + \bar{k}) & i \geq 0 \\ -(1-\phi) i + \frac{\kappa}{2} \left[\frac{i}{k + \bar{k}} \right]^2 (k + \bar{k}) & i < 0 \end{cases}$$

The object of interest is the partial irreversibility, $-(1-\phi)i$ for $i < 0$, and the quadratic adjustment cost with small κ and $\bar{k}$ makes the problem smooth. The case with partial irreversibility of capital, $\phi \in (0, 1)$, introduces investment risk as in Angeletos (2007), but within a framework featuring rich firm dynamics. Relative to models that abstract from investment risk (Buera and Shin, 2011; Moll, 2014), in the case with partial irreversibility self-financing is a poorer substitute for credit access.

The problem can be written recursively. The value of individuals before they default satisfies the following Hamilton-Jacobi-Bellman equation.

$$\rho V(a, k, z_j) = \max \Bigg\{ \rho V^{def}(0, 0, z_j), \max_{c,i} \Bigg[\frac{c^{1-\sigma}}{1-\sigma} + V_a \cdot \Big(\pi(k, z_j) + w + r_a \cdot a - c - i - \Phi(i, k) \Big) + V_k \cdot (i - \delta k) + \sum_{-j} \lambda_{z_j, -j} \big(V(a, k, z_{-j}) - V(a, k, z_j) \big) \Bigg] \Bigg\} \quad (1)$$

The maximization over consumption and investment is subject to $a_t \geq -\lambda k_t$. The variable $\lambda_{z_j,-j}$ is the intensity λ_z times the probability that the new draw of z from $G(z)$ is z_{-j}.

Similarly, the value of those in default satisfies the following equation.

$$
\begin{aligned}
\rho V^{def}(a,k,z_j) = \max_{c,i} \Bigg[& \frac{c^{1-\sigma}}{1-\sigma} + V_a^{def} \cdot \Big(\pi(k,z_j) + w + r_a \cdot a - c - i - \Phi(i,k) \Big) \\
& + V_k^{def} \cdot (i - \delta k) + \sum_{-j} \lambda_{z_{j,-j}} \big(V^{def}(a,k,z_{-j}) - V^{def}(a,k,z_j) \big) \\
& + \lambda_d \cdot \big(V(a,k,z_j) - V^{def}(a,k,z_j) \big) \Bigg]
\end{aligned} \tag{2}
$$

The maximization is subject to $a \geq 0$, as those in default are excluded from borrowing $(a < 0)$.

Banking Sector. The banking sector is competitive. Banks lend to entrepreneurs at r_b with loan-to-value constraint (requires $1/\lambda$ units of capital as collateral for each dollar of debt), and pay interest r_s to depositors. If an entrepreneur defaults, the bank liquidates the firm and gets back $\phi_b \cdot k$. That is, banks face a similar partial irreversible investment technology. to that of firms, but we allow the irreversibility to be potentially more binding for banks, reflecting limits to their ability to enforce credit contracts, $\phi_b \leq \phi$.

Given the deposit rate r_s and the distribution of firms over wealth, capital and productivity $G\left(a,k,z\right)$, the lending rate is set to guarantee that banks has zero profits. In particular,

$$
r_b = r_s + \lim_{\Delta \to 0} \frac{\int_{(a,k,z) \in \mathcal{I}_{default}} \left(\phi_b k + a\right) dG\left(a,k,z\right)}{B \Delta}
$$

where

$$
B \equiv - \int_{a<0} a dG\left(a,k,z\right)
$$

and $\mathcal{I}_{default}$ denotes the default set.

Stationary Competitive Equilibrium. A stationary competitive equilibrium is given by a joint distribution over individual state $G\left(a,k,z\right)$, investment and consumption policy functions $i\left(a,k,z\right)$ and $c\left(a,k,z\right)$, default decisions encoded in the default set $\mathcal{I}_{default}$, and prices (r_s, r_b, w) such that: (i) the investment and consumption policy functions $i\left(a,k,z\right)$ and $c\left(a,k,z\right)$, and the default decisions, are consistent with the values that solve the HJB equations describing the problem of individuals before and after default, i.e., equations (1) and (2); (ii) the labor and loan market clear; (iii) the borrowing rate r_b is consistent with zero profits by banks; (iv) the stationary joint distribution over individual state $G\left(a,k,z\right)$ solves the Kolmogorov forward equation define by the individual policy functions and the shock process for the productivity.

Planner's Problem. To measure deviations from an efficient benchmark, we compare our economy with the solution of a planner who faces the same partial irreversibility and a quadratic adjustment cost. Importantly, given that the planner controls the investment of a continuum of firms, all idiosyncratic uncertainty is averaged out. Therefore, there are no risk considerations for the planner.

This can be seen more clearly by considering a decentralization of the planner's problem in which risk neutral firms invest subject to the partial irreversibility of investment and quadratic adjustment costs; this decentralization is the algorithm that we use to solve the planner's problem.[2] The value function of a risk-neutral firm with state (k, z) solves the following Hamilton-Jacobi-Bellman equation

$$\begin{aligned}\rho V\left(k, z_j\right) &= \max_i \left(1-\beta\right) z^{\frac{1}{1-\beta}} \left(\frac{\beta}{w}\right)^{\frac{\beta}{1-\beta}} k^{\frac{\alpha}{1-\beta}} - i - \Phi\left(i, k\right) + \left(i - \delta k\right) V_k\left(k, z_j\right) \\ &\quad + \sum_{-j} \lambda_{z_{j,-j}} \big(V(a, k, z_{-j}) - V(a, k, z_j)\big).\end{aligned}$$

An equilibrium in this economy is given by a joint distribution over the individual state $G\left(k, z\right)$ and a wage that are consistent with labor market clearing and the Kolmogorov forward equation describing the evolution of the stationary distribution $G\left(k, z\right)$.

This decentralization makes it clear that risk considerations are absent from the investment decision of the planner. The capital invested in a state would be lower than in an economy with reversible investment due to the fact that the expected return to capital is lower, as the firm takes into account that capital will be kept in place when the productivity is lower when capital is inside of the inaction region, or due to the expected adjustment costs. In this economy, the equilibrium interest rate in this decentralization equals the discount rate $r_b = r_s = \rho$.

3 Calibration

We discuss a preliminary calibration of the benchmark economy, which we use in a quantitative exploration of the role of irreversible investment risk and credit frictions in development. The model is relatively parsimonious. There are two preference parameters: the discount rate ρ and the coefficient of relative risk aversion σ; five technological parameters: the capital and labor elasticities, α and β, respectively, the liquidation value of capital ϕ, and the the

[2] A direct solution of the planner's problem involves the solution of a Hamiltonian problem with the distribution of firms across capital and productivity levels as state. The solution of the decentralized equilibrium can more easily be implemented by modifying the codes of the benchmark model.

Parameter	**Target**	**Value**
Discount Rate (ρ)	Saving Rate, $i_s = 0.02$	0.09
Depreciation Rate (δ)	Standard	0.06
Risk Aversion (σ)	Standard	2.00
Production Function (α, β)	Buera et al. (2011)	$(0.3, 0.49)$
Liquidation Value (ϕ)	Kermani and Ma (2022)	0.35
Adjustment Costs $\left(\kappa, \bar{k}\right)$	Negligible (Avoid Jumps)	$(0.1, 0, 1)$
LTV Constraint (λ)	External Finance to GDP	0.75
Transition Out of Autarky (χ_{dn})	Dobbie et al. (2020)	0.143

Table 1: Calibration Summary Table

adjustment cost parameters κ and $\bar{k}$; and two parameters describing the extent of financial frictions: the loan-to-value constraint λ and the rate at which agents in default recover their access to credit χ_{dn}. The exogenous productivity process is described by seven parameters: five values for the productivities z_i, the arrival rate λ_z of a new productivity, and the probability of the lowest productivity p_1. Given p_1, the (equal) probability of drawing any other productivity $p_{-1} = (1 - p_1)/4$.

Tables 1 presents the calibrated values parametrizing preferences, technologies and external financial frictions. We briefly describe the targets and calibrated values.

The discount factor ρ is chosen to match an equilibrium saving rate of 0.02. As is standard in model with financial friction, a substaintially larger discount factor is required to match a given interest rate. The depreciation rate is chosen to match the value in the US National Income and Product Accounts. The risk aversion and the production function parameters are chosen to follow standard value in the literature.

We choose the liquidation value of capital, which is the key determinant of the degree of investment irreversibility, to match the evidence in Kermani and Ma (2022). They find that the liquidation value of fixed assets is 35% of the net book value in the average industry, which directly implies $\phi = 0.35$. Note also that for the present calibration, we assume $\phi = \phi_b$, so the bank liquidates the firm at the same price as the entrepreneur can liquidate the firm. We set the parameters on the quadratic adjustment cost functions to be relatively negligible values: the sole purpose of the quadratic portion of the adjustment cost is to smooth the numerical solution of the model. The loan-to-value constraint is set to so that the ratio of external finance (credit) to GDP in the model matches the ratio of private credit to GDP in the Indian economy in 2005. Finally, the rate of transition out of autarky is chosen to match the fact that in the United States Chapter 13 bankruptcy flagsate traditionally removed from a borrower's credit report after seven years, as discussed in Dobbie et al. (2020).

The parameter values describing the productivity process are shown in Table 2. We

	z_1	z_2	z_3	z_4	z_5
$z_1 = 0$	-	0.02	0.02	0.02	0.02
$z_2 = 0.25$	0.07	-	0.02	0.02	0.02
$z_3 = 0.50$	0.07	0.02	-	0.02	0.02
$z_4 = 1.00$	0.07	0.02	0.02	-	0.02
$z_5 = 2.00$	0.07	0.02	0.02	0.02	-

Table 2: Shock Intensity $\lambda_z = 0.15 \times p_{ij}$

choose the parameter values of this parsimoneous representation of the productivity process to match an autocorrelation coefficient and standard deviation of log productivity of 0.85 and 0.78, respectively, and an exit rate of 7%, which are in the middle range of the values reported by David et al. (2020).

4 Firm Behavior

Having calibrated our model, we now examine firm behavior and dynamics in this model. The key result of the section is that when investment is (partially) irreversible and entrepreneurs are risk averse, there will be slow adjustment dynamics, relative to the planner's solution. Thus, the strong self-financing dynamics present in standard models will be dampened substantially: in the following section we will analyze how this affects model aggregates.

To highlight firm dynamics in the model, we will focus in particular on the behavior of an entrepreneur with the highest possible productivity ($z = z_5$) but who starts with zero assets and capital ($a(0) = k(0) = 0$). At each moment in time, the entrepreneur does not know if she will stay productive, making investment a risky proposition. However, for clarity we will follow the path of assets and capital for a firm that happens to never get hit with a productivity shock, and thus stays at z_5.

This particular exercise focuses us on exactly the behavior that is at the core of dynamic models of financial frictions. The planner would like the most productive firms to hold the most capital. In traditional dynamic models with credit constraints, cash-poor entrepreneurs cannot fully cover the cost of capital to bring the firm to optimal scale, resulting in misallocation of capital relative to the first-best. However, a strong self-financing channel in traditional models means that entrepreneurs will quickly reach a level of capital near the planner's solution.

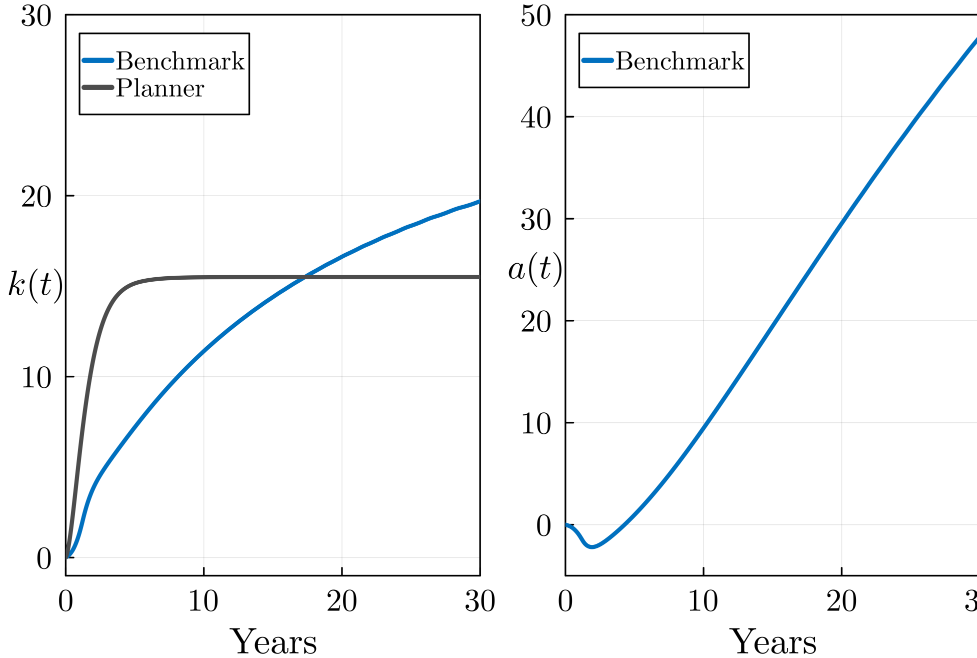

Figure 1: Paths of Capital and Assets for a Productive Firm ($z = z_5$)

Notes: This figure shows the path of assets and capital for a firm starting at $a = k = 0$ and which starts and stays at the maximum productivity ($z = z_5$). We solve for the steady state equilibrium under the main calibration of the model, and show paths for assets and capital. For comparison, the dashed line shows the path of capital under the planner's solution. In both cases, the firm does not know *ex ante* if it will stay at z_5, but happens *ex post* to never experience a productivity shock.

4.1 Investment Risk Leads to Slow Self-Financing

Figure 1 shows the path of assets and capital under this exercise for our main calibration, showing both the planner's solution and the constrained entrepreneur's solution. In each case, we solve the model in general equilibrium and show paths for entrepreneurs in the steady-state equilibrium; the interest rate facing entrepreneurs is thus somewhat different (lower) than the shadow cost of capital facing the planner.

The planner's solution yields a near-immediate jump to a socially optimal level of capital. The adjustment is slightly slowed by the quadratic adjustment cost, but this has only a small effect on the dynamics of capital.[3] Setting aside the small quadratic cost, an immediate jump makes sense since the optimal level of capital depends only on the firm's productivity, and there are no adjustment costs associated with increasing capital. Since the firm may face a negative productivity shock in the future and investment is partly irreversible, the planner assigns less capital than it would in a model without productivity shocks. This highlights the

[3]We included quadratic adjustment costs for computational reasons in order to smooth out the firm's problem, but intentionally set them to be small since they are not our economic focus.

importance of comparing to the planner's solution: by comparing to the planner's solution, we know that we are studying deviations from the first-best.

In contrast to the planner's capital jump dynamics, the constrained entrepreneur is very slow to self-finance. It takes 17 years before the entrepreneurs in our model reach the level of capital that the planner would hold. Note that this is not driven primarily by the credit constraint: capital stays well below the planner's solution even after the entrepreneur has enough assets to fully finance the firm to optimal scale.

What drives slow self-financing in our model, if not the credit constraint? The answer is investment risk. By the first welfare theorem, the planner's solution is equivalent to that of a profit-maximizing, risk-neutral entrepreneur. The planner simply maximizes the net present value of firm profits, under the appropriate social cost of capital and labor. The constrained entrepreneur is different because she is risk averse, and does not have access to insurance against risk. Irreversibility ($\phi = 0.35$) makes investment risky, which discourages investment.

Investment risk not only leads to low levels of investment, but also slows down the dynamic adjustment of capital after a positive productivity shock. Setting aside the credit constraint, the entrepreneur's optimal investment balances the high returns of investing in the firm against her risk aversion. As the entrepreneur becomes wealthier, her absolute risk aversion declines (we assume constant *relative* risk aversion), and so she is willing to invest more in a risky asset. Thus, as the entrepreneur stays productive and gets wealthy, her investment in the firm will approach and then exceed the optimal scale.[4]

This behavior is similar to well-understood behavior for a CRRA investor in a two-asset economy. In the two-asset economy, the CRRA investor puts a constant share of her wealth into the risky asset. Our entrepreneur's problem is different primarily because the firm faces decreasing returns to scale, so the optimal investment in the firm will asymptote to some scale, which turns out to be larger thant the scale the planner would choose.

As we will show next, the entrepreneur's solution in our model features much slower investment dynamics than in standard dynamic models with credit constraints. The reason is that *self-financing* is fast, but *self-insurance* is slow. In standard models without irreversibility, there is no investment risk and thus all assets are invested into the firm if the return to the firm is higher than the interest rate. Productive firms get high returns, accumulating

[4]In fact, in general equilibrium, a sufficiently wealthy entrepreneur will invest more capital into the firm than the planner would. This is not driven by precautionary saving: the entrepreneur has access to a risk-free asset, and so will never invest in the firm beyond the scale that equates the return on the risk-free asset and the expected return to the firm. Instead, the interest rate and wage are lower in the inefficient equilibrium than the corresponding shadow prices would be in the planner's allocation, and so a very wealthy entrepreneur will operate at a larger scale than the same firm would in the efficient economy.

capital quickly. Moreover, since all assets are invested into the firm, the entrepreneur will finance the firm to optimal scale even at low levels of assets.[5]

In contrast, in our model the entrepreneur will only invest a fraction of her wealth into the firm. This lowers returns, slowing investment dynamics. Moreover, she needs much more wealth before she fully finances the firm. Rather than saving her way out of a credit constraint, the entrepreneur is saving up until she is so wealthy that the investment risk stops mattering to her.

The slow dynamics of our model rely on two key ingredients: irreversibility and risk aversion. Irreversibility makes investment risky, while risk aversion makes entrepreneurs sensitive to this risk. We will show that removing either of these ingredients from our model would yield the fast self-financing that we see in standard models.

4.2 Self-Financing Is Fast When Investment Is Reversible

We next study firm's dynamic behavior in a version of the model without reversibility. We modify the calibration of the model by setting $\phi = 1$, which implies that the entrepreneur can resell capital at full price (setting aside the small quadratic adjustment costs). Figure 2 shows the path of assets and capital under this exercise for this alternate calibration, again showing both the planner's solution and the constrained entrepreneur's solution.

The results for the fully reversible calibration are quite different from those for the main calibration. Unsurprisingly, the planner's solution features higher levels of investment than the planner's solution in the main model. Without irreversibility, the planner's solution coincides with the optimal level of capital from a non-stochastic version of the model.

The entrepreneur's solution in the fully reversible calibration also features much stronger self-financing dynamics. Since investment is fully reversible, it is no longer risky (aside from the small quadratic adjustment costs). This gives us a model similar to standard dynamic models of credit constraints (Buera et al., 2015). The result is the fast self-financing dynamics that have previously documented in the literature: within 5 years, the constrained entrepreneur has already reached the level of capital that would have prevailed in the planner's solution. This shows that (partial) irreversibility is an essential ingredient for our model to yield slow investment dynamics.

[5] The firm will have an optimal scale if it faces decreasing returns to scale. In models with constant returns to scale, such as Moll (2014), productive firms will simply grow without bound until the firm's productivity falls below a threshold or the entrepreneur dies.

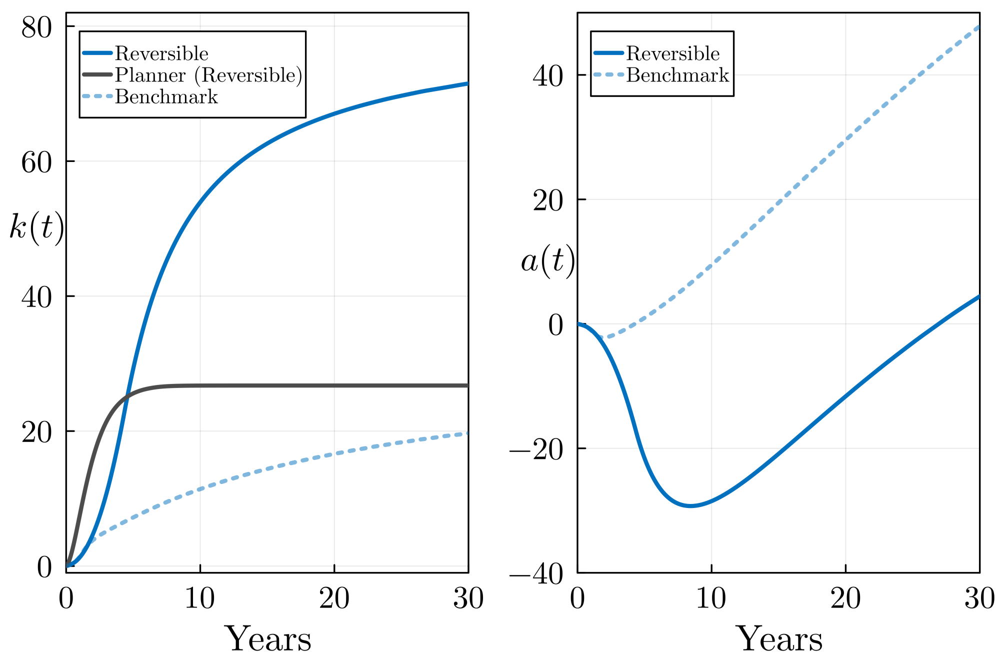

Figure 2: Paths of Assets and Capital for a Productive Firm: Calibration with Full Reversibility

Notes: This figure shows the path of assets and capital for a firm starting at $a = k = 0$ and which starts and stays at the maximum productivity ($z = z_5$), under an alternate calibration in which investment is fully reversible ($\phi = 1$). We solve for the steady state equilibrium and show paths for assets and capital. For comparison, the dashed line shows the path of capital under the planner's solution. In both cases, the firm does not know *ex ante* if it will stay at z_5, but happens *ex post* to never experience a productivity shock.

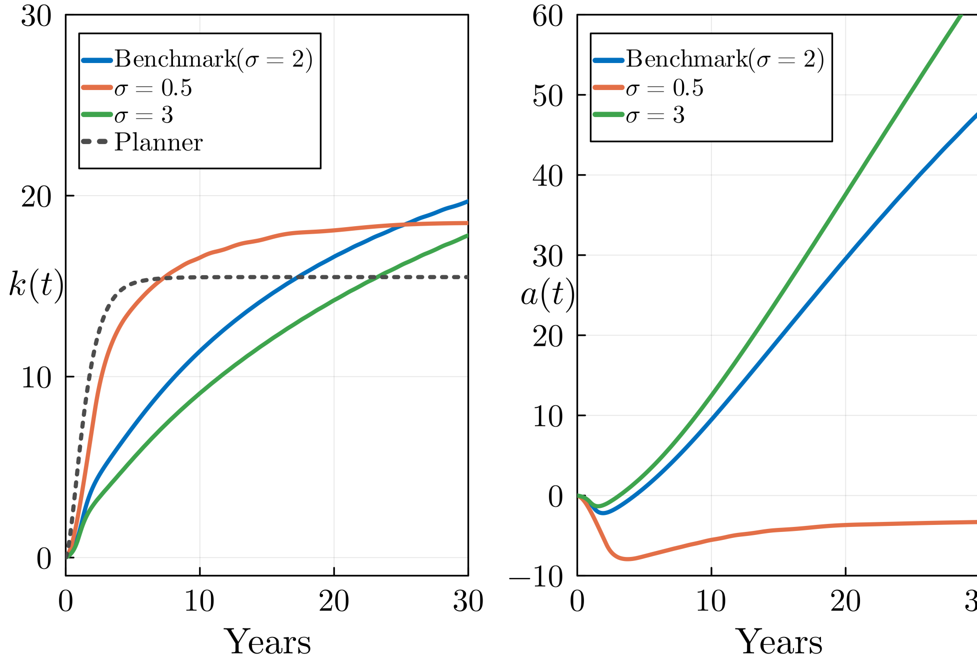

Figure 3: Paths of Assets and Capital for a Productive Firm: Calibration with Full Reversibility

Notes: This figure shows the path of assets and capital for a firm starting at $a = k = 0$ and which starts and stays at the maximum productivity ($z = z_5$), under alternate calibrations with different degrees of risk aversion. We solve for the steady state equilibrium and show paths for assets and capital. For comparison, the dashed line shows the path of capital under the planner's solution. In both cases, the firm does not know *ex ante* if it will stay at z_5, but happens *ex post* to never experience a productivity shock.

4.3 Self-Financing Is Fast When Firms Are Risk-Neutral

We next study how risk aversion affects the speed of self-financing in our model. We return to the main calibration ($\phi = 0.35$), but consider various degrees of risk aversion. Figure 3 shows the path of assets and capital under this exercise.

With less risk aversion ($\sigma = 0.5$), once again, we see that the entrepreneur's problem exhibits rapid growth in capital through debt financing. Given our use of time-separable utility, a low coefficient of relative risk aversion also implies a high intertemporal elasticity of substitution. The entrepreneur's problem becomes more similar to the planner's problem: the credit constraint limits capital in each period, but the entrepreneur does not find it too costly to delay consumption. As a result, the entrepreneur's solution for capital looks similar to the planner's solution.

With a higher degree of risk aversion ($\sigma = 3$), the opposite holds. The self-financing is slower than in the benchmark despite the fact that the entrepreneur holds more financial wealth than in the benchmark.

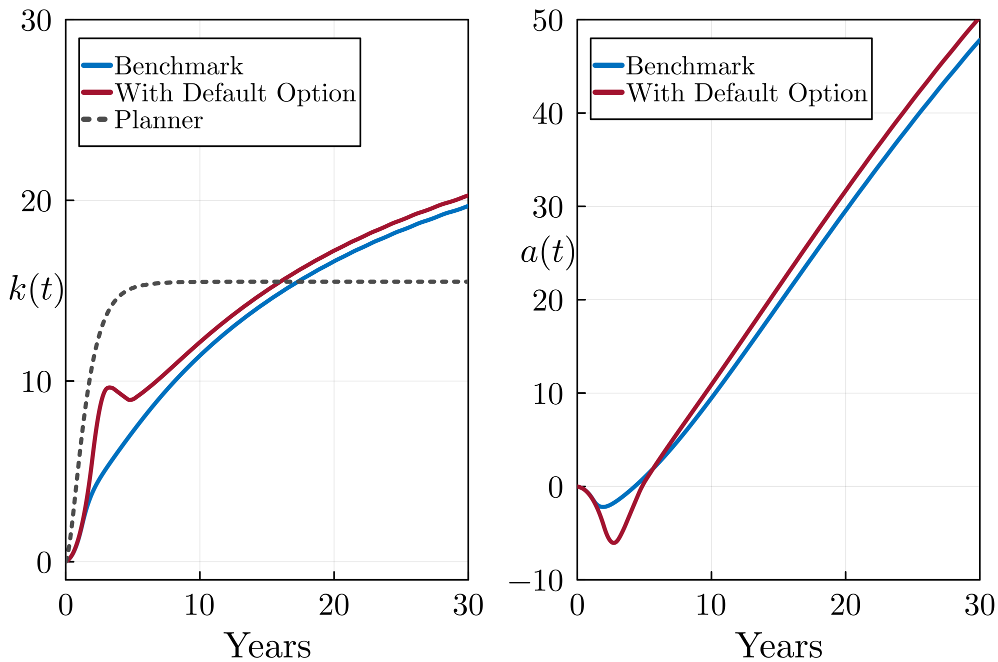

Figure 4: Paths of Assets and Capital for a Productive Firm: Calibration with Full Reversibility

Notes: This figure shows the path of assets and capital for a firm starting at $a = k = 0$ and which starts and stays at the maximum productivity ($z = z_5$), under an alternate model in which entrepreneurs can default on their debt. We solve for the steady state equilibrium and show paths for assets and capital. For comparison, the dashed line shows the path of capital under the planner's solution. In both cases, the firm does not know *ex ante* if it will stay at z_5, but happens *ex post* to never experience a productivity shock.

4.4 The Option to Default Counteracts Investment Risk

We now consider an alternate model specification, in which entrepreneurs can default on their debt. Upon default, they surrender their capital and are excluded from borrowing until stochastically reinstated. Because of the possibility of default, there is a credit spread between the interest rate on loans and the interest rate on deposits. 4 shows the path of assets and capital under this exercise.

The default option partly insures entrepreneurs from their loss of high productivity, and entreprenerus with a small capital stock are more willing to take out loans to invest than in the benchmark where default was not allowed. The poor but productive entrepreneur borrow to rapidly grow her capial stock initially, much faster than in the benchmark. Once the capital stock reaches a certain level, the entrepreneur disinvests slightly to pay off her high-interest debt, before investing further through self-financing.

5 General Equilibrium Effects of Investment Risk and Financial Frictions

We now study the effects of investment risk and financial frictions on the steady-state equilibrium of the model. We focus first on our main calibration, and then on the alternative calibrations that we studied in the previous section: full reversibility ($\phi = 1$), low risk aversion ($\sigma = 0.5$), elevated risk aversion ($\sigma = 3$), and a version of the model with the option to default. Throughout, we will compare the (inefficient) equilibrium of the model to the planner's solution.

5.1 Effects on Output, Capital, and Welfare

We begin by studying how investment risk affects key aggregates in the steady state of the model. In particular, we study aggregate output $Y := \mathbb{E}[y]$, aggregate capital $K := \mathbb{E}[k]$, aggregate productivity $Z := Y/\left(K^{\alpha}L^{\beta}\right)$, and steady-state average welfare, which we define momentarily. For output, capital, and productivity, we divide each of these by its efficient counterpart in the steady state of the planner's solution, which we denote with a star (e.g. Y/Y^* is the ratio of steady-state output in the model to the planner's solution).

Welfare Losses. To study the effect of financial frictions on welfare in our model, we must express our results in units of consumption. We use an approach similar to compensating/equivalent variation. In particular, we ask what decline in consumption under the planner's solution would be necessary to make the entrepreneur indifferent between the steady-state of the frictional equilibrium and the steady-state of the planner's solution. More precisely, steady-state consumption in the planner's solution is defined as aggregate output minus depreciation and investment costs:

$$C_p := \int \left(zk^{\alpha}l^{\beta} - \delta k - i - \Phi(i,k)\right) d\Omega.$$

We then define g to equate average utility in the frictional equilibrium with utility in the planner's solution (with consumption multiplied by $1-g$)

$$\int \frac{(c)^{1-\sigma}}{1-\sigma} dG = \frac{\left(C_p(1-g)\right)^{1-\sigma}}{1-\sigma}$$

$$-g = \left[\frac{\int c^{1-\sigma} dG}{C_p^{1-\sigma}}\right]^{\frac{1}{1-\sigma}} - 1 \qquad (3)$$

	Benchmark	$\phi = 1$	$\sigma = 0.5$	$\sigma = 3$	With Default
Y/Y^*	0.92	1.10	1.03	0.87	0.9
K/K^*	1.00	1.61	1.72	0.86	0.89
Z/Z^*	0.92	0.94	0.88	0.91	0.93
Welfare Losses $(-g)$	19%	13%	6%	25%	21%

Table 3: Aggregates in Steady-State Equilbrium

Notes: This table shows how frictions affect the model's aggregate in steady state. In the first three rows, we compute the ratio between the (inefficient) equilibrium value of an aggregate and the value of that aggregate in the planner's solution, using X^* to denote the value of X in the planner's solution. We compute aggregate output Y, aggregate capital $K := \mathbb{E}[k]$, and aggregate productivity $Z := Y/\left(K^\alpha L^\beta\right)$. In the fourth row, we compute the consumption-equivalent welfare losses, as defined in Equation 3. Each column studies a different calibration of the model. The first column shows the main calibration of the model, as described in Section 3. The second column shows a recalibration of the model where $\phi = 1$, implying full reversibility of investment, but all other parameters are as in the main calibration. The third column is the same as the main calibration except the risk aversion parameter, σ, is set to 0.5, making entrepreneurs closer to risk-neutral. The fourth column instead raises risk aversion to $\sigma = 3$. The fifth column uses the main calibration, but introduces the option for firms to default on their loans.

A few things are worth noting here. First, we are comparing welfare across two steady-states, rather than computing the welfare from starting at one steady-state and then suddenly imposing/removing frictions. Thus, our measure will not capture any costs or benefits associated with the transition path. Second, as in Buera and Shin (2011), the welfare losses from financial frictions reflect both a lack of consumption insurance and lower aggregate consumption than would prevail in the planner's solution. This lowered aggregate consumption will come in part from misallocation of capital across firms, and may potentially also come from a lowered level of aggregate capital (quantitatively, we find that aggregate capital can in fact be higher than in the planner's solution at low levels of risk aversion).

The results are in Table 3. In the main calibration of the model, frictions lower aggregate output by 8%. This is driven entirely by a reduction in aggregate productivity of 8%, as the aggregate capital stock is roughly unchanged in the main calibration. This is because two forces cancel each other out: investment risk lowers capital, but precautionary savings raises it. The reduction in average welfare is 19%, which is substantially larger than the reduction in aggregate output. This difference reflects the fact that the planner not only achieves higher output than the inefficient equilibrium, but also implements complete risk sharing.

In Section 4, we showed that irreversibility and risk aversion were essential ingredients for our model's slow adjustment dynamics, because they made investment risky. To study how investment risk affects the model, as opposed to just credit constraints, we can recalibrate the model to have either full reversibility ($\phi = 1$) or low risk aversion ($\sigma = 0.5$). The results are in the second and third columns of Table 3. In both cases, with investment risk

	$\sigma = 0.5$	$\sigma = 2$	$\sigma = 3$	$\sigma = 4$	$\sigma = 5$	$\sigma = 10$
Y/Y^*	1.03	0.92	0.87	0.83	0.81	0.75
K/K^*	1.72	1.00	0.86	0.76	0.70	0.58
Z/Z^*	0.88	0.92	0.91	0.91	0.90	0.89
Welfare Losses $(-g)$	6%	19%	25%	28%	31%	35%

Table 4: Aggregates in Steady-State Equilbrium (Varying Risk Aversion)

Notes: This table shows how frictions affect the model's aggregate in steady state, for varying levels of risk aversion. In the first three rows, we compute the ratio between the (inefficient) equilibrium value of an aggregate and the value of that aggregate in the planner's solution, using X^* to denote the value of X in the planner's solution. We compute aggregate output Y, aggregate capital $K := \mathbb{E}[k]$, and aggregate productivity $Z := Y/\left(K^\alpha L^\beta\right)$. In the fourth row, we compute the consumption-equivalent welfare losses, as defined in Equation 3. Each column studies a different calibration of the model. Each column shows the main calibration of the model, as described in Section 3, but with a different coefficient of relative risk aversion, σ.

minimized, the precautionary savings channel dominates. As a result, the full reversibility case has 61% more capital in steady-state than the planner would hold, while the low risk aversion case has 72% more capital. As a result, steady-state output is actually higher in these calibrations than in the planner's solution, although aggregate productivity is lower (by 6% for the full-reversibility case, and by 12% in the low risk aversion case.

The third column shows results for slightly higher risk aversion, $\sigma = 3$. At this level of risk aversion, investment risk dominates precautionary savings, and aggregate capital is reduced. We will explore how different levels of risk aversion interact with investment risk in the next subsection.

5.2 Investment Risk Lowers Output, Capital, and Welfare at High Risk Aversion

At modest levels of risk aversion, our model does not necessarily deliver lowered levels of steady-state capital, relative to the planner's solution. However, as we saw in the previous subsection, raising the level of risk aversion will lower aggregate capital. We now explore more thoroughly how risk aversion interacts with investment risk.

In Table 4, we vary the coefficient of relative risk aversion σ, and see how it affects aggregates in the steady-state equilibrium. We find that at higher levels of risk aversion, the investment risk channel dominates precautionary savings, and the capital stock falls dramatically. At the highest risk aversion we consider, $\sigma = 10$, we find that aggregate output is 25% lower than in the planner's solution, capital is 42% lower, aggregate productivity is 11% lower, and there is a 35% welfare loss due to financial frictions. The losses in output, capital and welfare appear to be monotone in risk aversion, while aggregate productivity appears to be decreasing in risk aversion for $\sigma > 2$, after initially increasing between $\sigma = 0.5$ and $\sigma = 2$.

These results suggest that the level of risk aversion is quite important for understanding the negative effects of investment risk. Although $\sigma = 10$ is towards the high end of the risk aversion parameters typically used in the macroeconomics literature, it is not necessarily an unreasonable value for the risk aversion of entrepreneurs in a developing country. For one, entrepreneurs in developing countries may be particularly sensitive to risk due to subsistence concerns: it is very costly to reduce consumption when the margin of adjustment is consumption of food or other necessities. Moreover, there is some evidence for $\sigma = 10$ even in developed country contexts. For example, Best et al. (2019) estimate an elasticity of intertemporal substitution of 0.1 using mortgage data from the United Kingdom; for a CRRA utility function, this implies a coefficient of relative risk aversion of $\sigma = 10$.

5.3 Investment Risk Generates Dispersion in *Ex Ante* Returns

Misallocation of capital can be understood in terms of marginal products: there is misallocation when the marginal product of capital differs across firms, and there is underinvestment when the average marginal product of capital is too high. A great deal of microeconomic evidence suggests that returns to capital are high and dispersed in less developed economies, which poses problems for models with a strong self-financing channel. We now show that our model, by adding investment risk into the model, can generate meaningful dispersion in returns to capital.

Our model features an important complication: even in the efficient economy, there will be some dispersion in returns to capital. Because the adjustment cost is not continuously differentiable[6], the planner's solution to our model features inaction regions. A firm that starts with a high level of capital and then draws a low productivity will face a region in which the optimal strategy is zero investment, where the returns to capital are too low to justify investment, but too high to justify liquidating capital at a discount. In this inaction region, the planner's first order condition is a pair of inequalities rather than an equation, and thus dispersion in returns need not imply inefficiency.

Even if our adjustment costs were continuously differentiable (e.g. quadratic costs only), the planner's solution would still admit dispersion in returns to capital, depending on how returns are measured. With adjustment costs, the level of capital has dynamic implications, and so investment decisions must account not just for the effect of capital on immediate profits, but on the entire stream of future profits. Investment in a high productivity firm

[6]The marginal cost of capital jumps discretely at zero investment: the marginal cost is higher when the firm is buying capital (full price) than when it is liquidating capital (discounted by ϕ).

	Planner	$\sigma = 0.5$	$\sigma = 2$	$\sigma = 3$	$\sigma = 4$	$\sigma = 5$	$\sigma = 10$
$\mathbb{E}\,[\text{MPK}]$	0.15	0.12	0.16	0.18	0.20	0.21	0.24
Median MPK	0.13	0.11	0.16	0.18	0.19	0.21	0.24
SD (MPK)	0.13	0.07	0.12	0.14	0.16	0.16	0.17

Table 5: Monthly Returns to Capital in Steady-State Equilbrium

Notes: This table shows how frictions affect the model's distribution of returns to capital, as defined in the text in Section 5.3. The first column shows the distribution of returns under the planner's solution, while the remaining columns show the distribution of returns for firms under the (inefficient) steady-state equilibrium with frictions. We compute mean returns $\mathbb{E}\,[\text{MPK}]$, median returns, and the standard deviation of returns SD (MPK). We focus on the main calibration of the model, as described in Section 3, but vary the risk aversion across columns.

may yield high returns today, but the firm's productivity will eventually fall, and the firm will have to pay some adjustment cost to wind down the investment. Thus, in the planner's solution, we would expect that short-run returns to capital will be higher at more productive firms, since in the long-run those firms will become less productive and the capital will have to be disinvested.

For simplicity, in the following analysis, we measure returns as the firm's marginal product of capital: $\frac{\alpha z l^{\beta}}{k^{1-\alpha}}$. Since this is always zero if $z = 0$, and is undefined for $k = 0$, we only compute the MPK for firms with $z > 0$ and $k > 0$. We can benchmark the level and dispersion in returns that we measure against those same values under the planner's solution, to better understand the degree of inefficiency.

In Table 5, we show the mean, median, and standard deviation of returns in steady state. Each column shows a different calibration of the model, with the first column showing the planner's solution for reference. We show mean and median returns to capital, as well as the standard deviation of returns to capital.

Following the previous subsection, we vary the level of risk aversion to see how risk aversion interacts with investment risk to affect the distribution of returns to capital. Note that by varying only risk aversion, we do not affect the (steady-state) planner's solution. Since consumption is constant in the steady-state, the only preference parameter relevant to the planner's solution is the discount rate, because it determines the shadow interest-rate.

The results broadly mirror the previous results for aggregates. First, note that even in the planner's solution to this model, there is dispersion in the marginal product of capital, for the reasons discussed above. Second, the results for the level and dispersion of MPK mirror our earlier results: at low levels of risk aversion, the returns to capital are actually lower than in the planner's problem, due to precautionary savings (this is the mirror to our result on the level of capital for $\sigma = 0.5$). As risk aversion rises, the mean and median MPK rise, reaching a level of 24% annual returns for $\sigma = 10$. The standard deviation of returns also rises as the risk aversion rises. These results suggest that investment risk could play a role in explaining high and dispersed returns to capital in developing countries.

6 Conclusion

In this paper, we explored a model of financial frictions with risk averse entrepreneurs, lack of insurance, and partially irreversible investment. The addition of partial irreversibility to an otherwise standard model of financial frictions creates investment risk: because capital will stick around, the entrepreneur must consider the firm's future productivity over the lifetime of the capital. Because of risk aversion and lack of insurance, this leads to underinvestment at sufficient levels of risk aversion, since low productivity states of the world feature both low returns and low consumption.

This mechanism generates not just underinvestment but also a rich set of dynamics and heterogeneity. The critical mechanism is that self-financing is replaced with self-insurance. Entrepreneurs do not need to be very wealthy in order to self-finance the firm out of their own assets. But with investment risk, the constraint is not insufficient assets to finance the firm, but rather insufficient wealth to self-insure against potential productivity shocks. Self-insurance is slow: it takes a long time to accumulate enough wealth to self-insure against shocks, especially since the entrepreneur is investing in low-return savings rather than the high-return firm. This also generates a link between inequality and misallocation: entrepreneurs with identical firms but different levels of wealth will invest different amounts in their firm, and wealthier entrepreneurs will thus have lower returns on the margin.

We explored these forces in a preliminary calibration of our model. We found that at high levels of risk aversion, our model can deliver a substantial reduction in aggregate output, capital, productivity, and welfare. Moreover, our model can generate high and dispersed returns. This suggests that investment risk is a promising candidate to explain high and dispersed returns to capital in developing countries, as well as pervasive low levels of capital in these economies.

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